FAST_LIO:
理论与复现

参考: FAST-LIO论文解读与详细公式推导 - 知乎

2025.11

目 录

1 背景与系统概览

1.1 背景

主要关注:

[1] *W. Xu and F. Zhang, "FAST-LIO: A Fast, Robust LiDAR-Inertial Odometry Package by Tightly-Coupled Iterated Kalman Filter," in IEEE Robotics and Automation Letters, vol. 6, no. 2, pp. 3317-3324, April 2021.*

[2] *W. Xu, Y. Cai, D. He, J. Lin and F. Zhang, "FAST-LIO2: Fast Direct LiDAR-Inertial Odometry," in IEEE Transactions on Robotics, vol. 38, no. 4, pp. 2053-2073, Aug. 2022.*

同步定位与建图 (Simultaneous localization and mapping, SLAM) 对于移动机器人，比如无人机[**unmanned aerial vehicles (UAVs)**] 来说，是重要的前提条件。

视觉（惯性）里程计【Visual (-inertial) odometry **(VO/VIO)**】:

- 轻量化且廉价，能提供丰富的 RGB 信息
- 然而：
  - **缺少直接的深度测量，**为了重建 3D 场景，需要大量的**计算资源**
  - **对光照十分敏感**

激光雷达 (Light detection and ranging, **LiDAR**) **传感器可以克服所有这些障碍。**

1.2 问题与解决方案

问题	解决方案
1 在**复杂环境中**没有明显特征时，基于 LiDAR 的方案很容易**退化(degenerate)**.	1 采用**紧耦合迭代卡尔曼滤波器（tightly-coupled iterated Kalman filter）**来融合 LiDAR 特征点和 IMU 测量。

松耦合策略（Loosely-coupled approach）：

- **首先**通过雷达点云匹配获得位姿估计，**再**将位姿估计与IMU测量融合。
- 雷达的估计误差大时，无能为力。

紧耦合策略（Tightly-coupled approach）：

- 直接将 LiDAR 特征点 (而非匹配后得出的位姿) 与IMU测量融合以得到位姿。
- **互补校正效果**：IMU 校正雷达点云的畸变，雷达点云校正 IMU 的漂移。

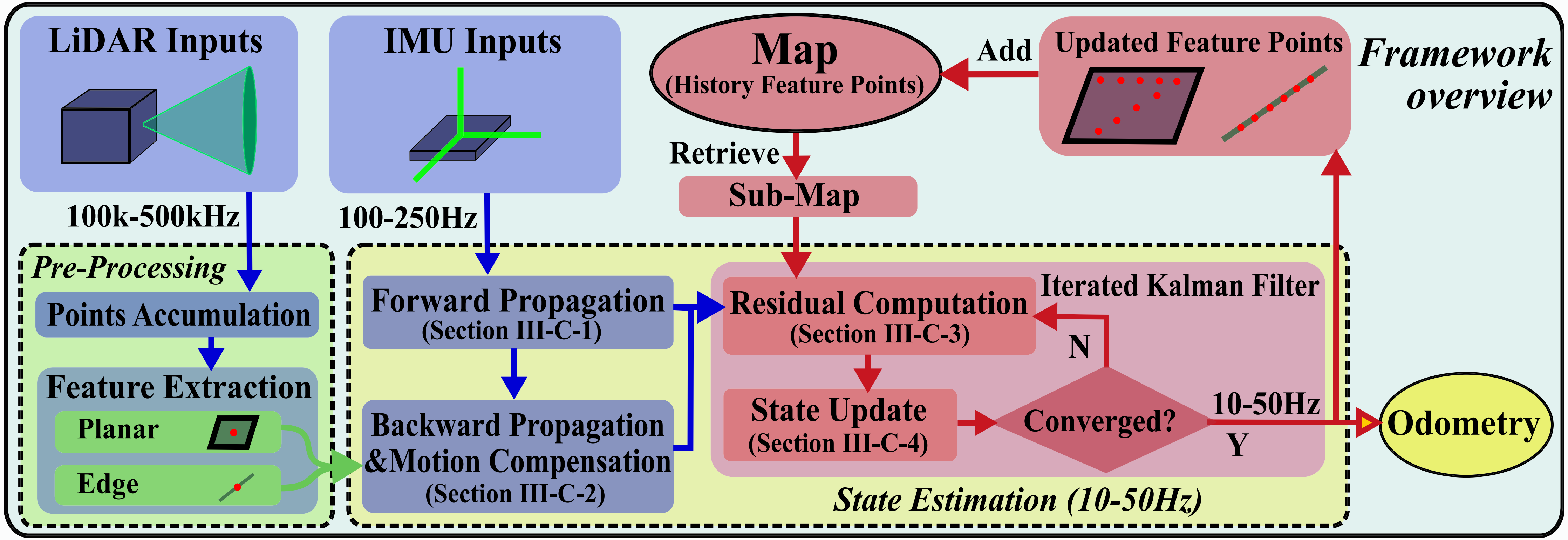
1.2 问题与解决方案

问题

1. 在**复杂环境中**没有明显特征时，基于 LiDAR 的方案很容易**退化(degenerate)**.
2. 将大量特征点和IMU测量进行紧耦合**需要大量的算力。**
3. 一次扫描内的雷达点总是在不同时刻被采集到的，这会导致**运动畸变（motion distortion）**并降低点云匹配的质量。

解决方案

1. 采用**紧耦合迭代卡尔曼滤波器（tightly-coupled iterated Kalman filter）**来融合 LiDAR 特征点和 IMU 测量。
2. 提出了一种**新的卡尔曼增益计算公式** 并证明其与传统卡尔曼增益公式的等价性。
3. 提出一种**反向传播（back-propagation）过程**以补偿运动畸变。


LiDAR Inputs
IMU Inputs
Map
(History Feature Points)
Add
Updated Feature Points
Framework overview
Retrieve
Sub-Map
100k-500kHz
100-250Hz
Pre-Processing
Points Accumulation
Feature Extraction
Planar
Edge
Forward Propagation
(Section III-C-1)
Backward Propagation
&Motion Compensation
(Section III-C-2)
Residual Computation
(Section III-C-3)
Iterated Kalman Filter
N
State Update
(Section III-C-4)
Converged?
10-50Hz
Y
Odometry
State Estimation (10-50Hz)

2 状态估计

连续动力学:

改进：重力对齐

$${}^{G}\dot{\boldsymbol{p}}_I = {}^{G}\boldsymbol{v}_I$$

世界坐标系（此工作中为 **IMU 第一帧**）下的 IMU 位置和速度

$${}^{G}\dot{\boldsymbol{v}}_I = {}^{G}\boldsymbol{R}_I\left(\underbrace{\boldsymbol{a}_m}_{\text{测量}} - \underbrace{\boldsymbol{b}_a}_{\text{零偏}} - \underbrace{\boldsymbol{n}_a}_{\text{噪声}}\right) + {}^{G}\boldsymbol{g}$$

世界坐标系下的 IMU 加速度

$${}^{G}\dot{\boldsymbol{g}} = \boldsymbol{0}$$

世界坐标系下的重力向量

$${}^{G}\dot{\boldsymbol{R}}_I = {}^{G}\boldsymbol{R}_I\left\lfloor \underbrace{\boldsymbol{\omega}_m}_{\text{测量}} - \underbrace{\boldsymbol{b}_\omega}_{\text{零偏}} - \underbrace{\boldsymbol{n}_\omega}_{\text{噪声}} \right\rfloor_\wedge$$

$\lfloor \boldsymbol{a} \rfloor_\wedge$ 表示由向量 $\boldsymbol{a}$ 所得的叉积因子，$\lfloor \boldsymbol{a} \rfloor_\wedge = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$

$$\left.\begin{aligned} \dot{\boldsymbol{b}}_\omega &= \boldsymbol{n}_{b\omega} \\ \dot{\boldsymbol{b}}_a &= \boldsymbol{n}_{ba} \end{aligned}\right\}$$

IMU 零偏（bias），建模为具有高斯噪声的随机游走过程

连续动力学:

$$
{}^{G}\dot{\boldsymbol{p}}_{I} = {}^{G}\boldsymbol{v}_{I}
$$

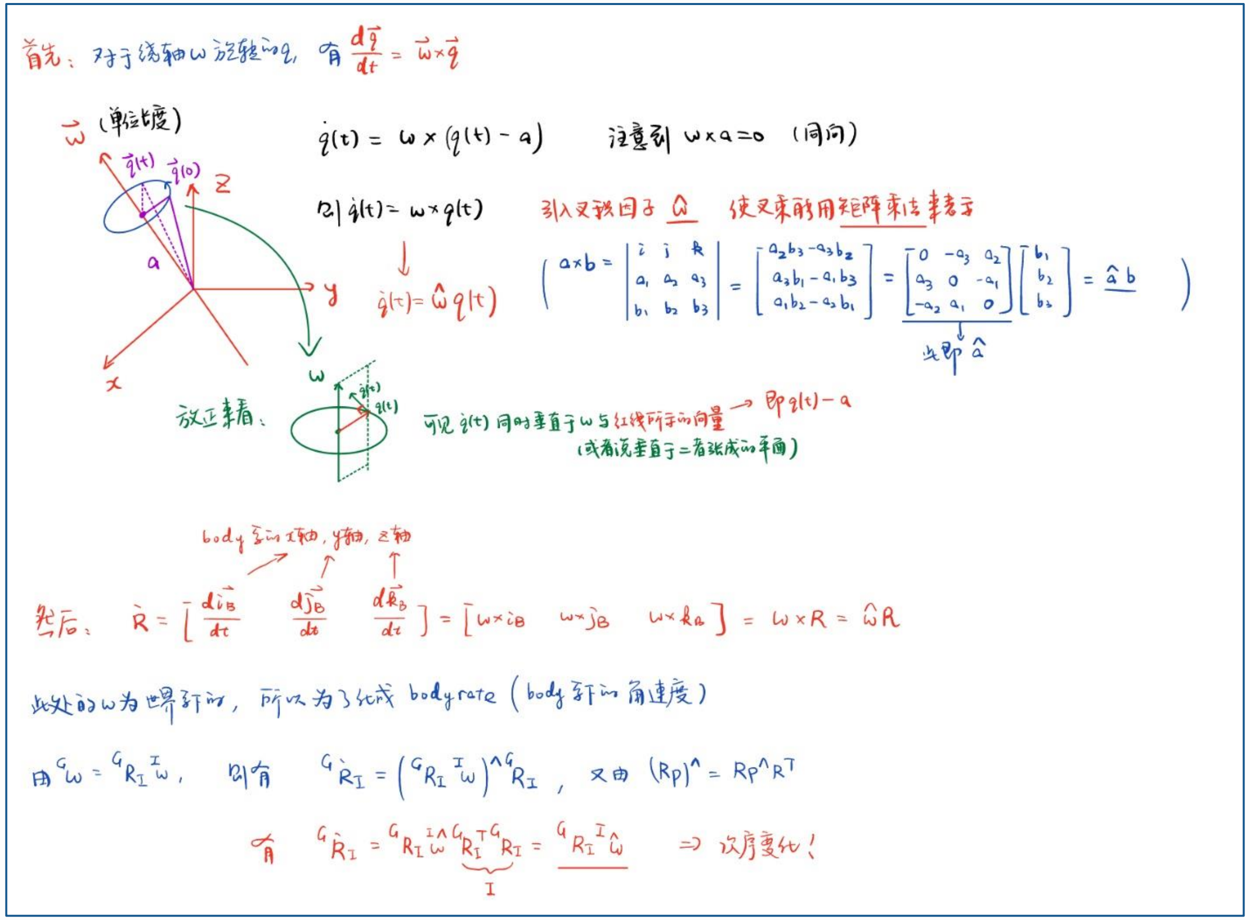
$$
{}^{G}\dot{\boldsymbol{v}}_{I} = {}^{G}\boldsymbol{R}_{I}\left(\overset{\text{测量}}{\boldsymbol{a}_{m}} - \overset{\text{零偏}}{\boldsymbol{b}_{a}} - \overset{\text{噪声}}{\boldsymbol{n}_{a}}\right) + {}^{G}\boldsymbol{g}
$$

$$
{}^{G}\dot{\boldsymbol{g}} = \boldsymbol{0}
$$

$$
{}^{G}\dot{\boldsymbol{R}}_{I} = {}^{G}\boldsymbol{R}_{I}\left\lfloor \overset{\text{测量}}{\boldsymbol{\omega}_{m}} - \overset{\text{零偏}}{\boldsymbol{b}_{\omega}} - \overset{\text{噪声}}{\boldsymbol{n}_{\omega}} \right\rfloor_{\wedge}
$$

$$
\dot{\boldsymbol{b}}_{\omega} = \boldsymbol{n}_{b\omega}
$$

$$
\dot{\boldsymbol{b}}_{a} = \boldsymbol{n}_{ba}
$$



连续 -> 离散:

$$\mathbf{x}_{i+1} = \mathbf{x}_i \boxplus \left(\Delta t \mathbf{f}(\mathbf{x}_i, \mathbf{u}_i, \mathbf{w}_i)\right)$$

$$^G\dot{\mathbf{p}}_I = {}^G\mathbf{v}_I$$

$$^G\dot{\mathbf{v}}_I = {}^G\mathbf{R}_I\left(\mathbf{a}_m - \mathbf{b}_a - \mathbf{n}_a\right) + {}^G\mathbf{g}$$

$$^G\dot{\mathbf{g}} = \mathbf{0}$$

$$^G\dot{\mathbf{R}}_I = {}^G\mathbf{R}_I\lfloor \boldsymbol{\omega}_m - \mathbf{b}_\omega - \mathbf{n}_\omega \rfloor_\wedge$$

$$\dot{\mathbf{b}}_\omega = \mathbf{n}_{b\omega}$$

$$\dot{\mathbf{b}}_a = \mathbf{n}_{ba}$$

其中 $\mathcal{M} = SO(3)\times\mathbb{R}^{15}, \dim(\mathcal{M}) = 18$

$$\mathbf{x} \doteq \begin{bmatrix} {}^G\mathbf{R}_I^T & {}^G\mathbf{p}_I^T & {}^G\mathbf{v}_I^T & \mathbf{b}_\omega^T & \mathbf{b}_\mathrm{a}^T & {}^G\mathbf{g}^T \end{bmatrix}^T \in \mathcal{M}$$

$$\mathbf{u} \doteq \begin{bmatrix} \boldsymbol{\omega}_m^T & \mathbf{a}_m^T \end{bmatrix}^T, \mathbf{w} \doteq \begin{bmatrix} \mathbf{n}_\omega^T & \mathbf{n}_\mathrm{a}^T & \mathbf{n}_{\mathrm{b}\omega}^T & \mathbf{n}_{\mathrm{ba}}^T \end{bmatrix}^T$$

一阶欧拉离散

$$\mathbf{f}(\mathbf{x}_i, \mathbf{u}_i, \mathbf{w}_i) = \begin{bmatrix} \boldsymbol{\omega}_{m_i} - \mathbf{b}_{\omega_i} - \mathbf{n}_{\omega_i} \\ {}^G\mathbf{v}_{I_i} \\ {}^G\mathbf{R}_{I_i}\left(\mathbf{a}_{m_i} - \mathbf{b}_{a_i} - \mathbf{n}_{a_i}\right) + {}^G\mathbf{g}_i \\ \mathbf{n}_{b\omega_i} \\ \mathbf{n}_{ba_i} \\ \mathbf{0}_{3\times1} \end{bmatrix}$$

$$\boxplus : \mathcal{M}\times\mathbb{R}^n \to \mathcal{M}; \quad \boxminus : \mathcal{M}\times\mathcal{M} \to \mathbb{R}^n$$

$$\mathcal{M} = SO(3) : \mathbf{R} \boxplus \mathbf{r} = \mathbf{R}\mathrm{Exp}(\mathbf{r}); \quad \mathbf{R}_1 \boxminus \mathbf{R}_2 = \mathrm{Log}(\mathbf{R}_2^\mathrm{T}\mathbf{R}_1)$$

$$\mathcal{M} = \mathbb{R}^n : \quad \mathbf{a} \boxplus \mathbf{b} = \mathbf{a} + \mathbf{b}; \quad \mathbf{a} \boxminus \mathbf{b} = \mathbf{a} - \mathbf{b}$$

目标: (1) 计算每个 IMU 测量到达时刻的状态，为雷达点去畸变作准备；(2) 获得 x_k 的先验估计

定义误差： (注意下标：i 是 IMU 测量的 ID，k 是雷达帧 (一次扫描 scan) 的 ID)

$$\tilde{\mathbf{x}}_{k-1} \doteq \mathbf{x}_{k-1} \boxminus \bar{\mathbf{x}}_{k-1} = \begin{bmatrix} \delta\boldsymbol{\theta}^T & {}^G\tilde{\boldsymbol{p}}_I^T & {}^G\tilde{\boldsymbol{v}}_I^T & \tilde{\boldsymbol{b}}_\omega^T & \tilde{\boldsymbol{b}}_a^T & {}^G\tilde{\boldsymbol{g}}^T \end{bmatrix}^T \in \mathbb{R}^{18} \qquad \delta\boldsymbol{\theta} = \mathrm{Log}({}^G\bar{\boldsymbol{R}}_I^T\,{}^G\boldsymbol{R}_I) \in \mathbb{R}^3$$

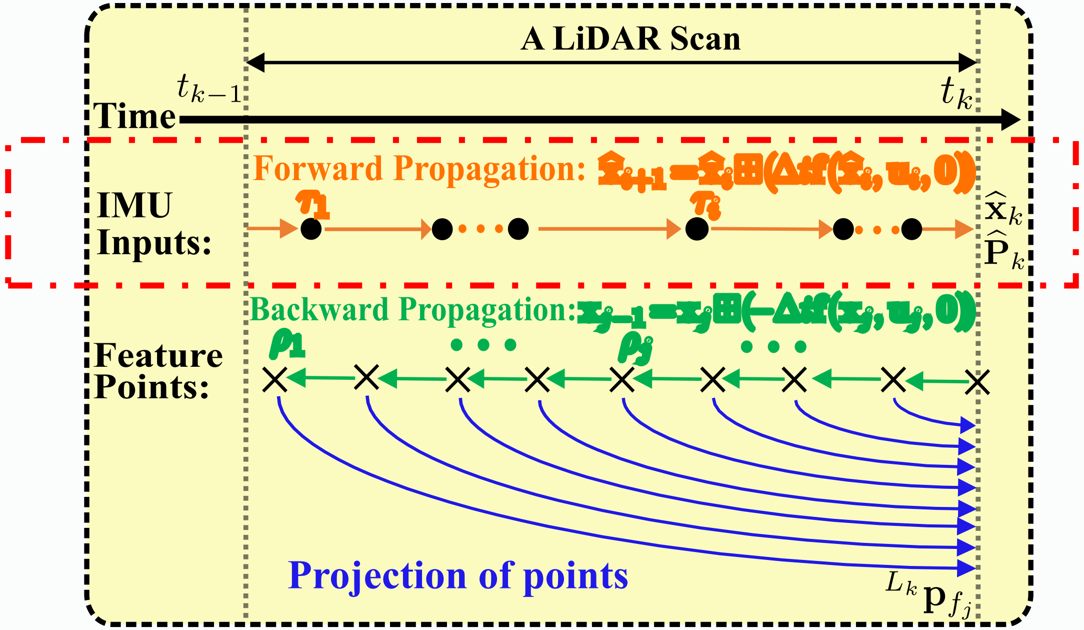
误差动力学：

$$\hat{\mathbf{x}}_{i+1} = \hat{\mathbf{x}}_i \boxplus (\Delta t \boldsymbol{f}(\hat{\mathbf{x}}_i, \boldsymbol{u}_i, \mathbf{0})); \hat{\mathbf{x}}_0 = \bar{\mathbf{x}}_{k-1}.$$

$$\begin{aligned}\tilde{\mathbf{x}}_{i+1} &= \mathbf{x}_{i+1} \boxminus \hat{\mathbf{x}}_{i+1} \\ &= \left(\mathbf{x}_i \boxplus \Delta t \boldsymbol{f}(\mathbf{x}_i, \boldsymbol{u}_i, \boldsymbol{w}_i)\right) \boxminus \left(\hat{\mathbf{x}}_i \boxplus \Delta t \boldsymbol{f}(\hat{\mathbf{x}}_i, \boldsymbol{u}_i, \mathbf{0})\right) \\ &= \left((\hat{\mathbf{x}}_i \boxplus \tilde{\mathbf{x}}_i) \boxplus \Delta t \boldsymbol{f}(\mathbf{x}_i, \boldsymbol{u}_i, \boldsymbol{w}_i)\right) \boxminus \left(\hat{\mathbf{x}}_i \boxplus \Delta t \boldsymbol{f}(\hat{\mathbf{x}}_i, \boldsymbol{u}_i, \mathbf{0})\right)\end{aligned}$$

定义 $\boldsymbol{g}(\tilde{\mathbf{x}}_i, \boldsymbol{w}_i) = \boldsymbol{f}(\mathbf{x}_i, \boldsymbol{u}_i, \boldsymbol{w}_i)\Delta t = \boldsymbol{f}(\hat{\mathbf{x}}_i \boxplus \tilde{\mathbf{x}}_i, \boldsymbol{u}_i, \boldsymbol{w}_i)\Delta t$

则有 $\tilde{\mathbf{x}}_{i+1} = \underbrace{\left((\hat{\mathbf{x}}_i \boxplus \tilde{\mathbf{x}}_i) \boxplus \boldsymbol{g}(\tilde{\mathbf{x}}_i, \boldsymbol{w}_i)\right) \boxminus \left(\hat{\mathbf{x}}_i \boxplus \boldsymbol{g}(\mathbf{0}, \mathbf{0})\right)}_{\boldsymbol{G}(\tilde{\mathbf{x}}_i, \boldsymbol{g}(\tilde{\mathbf{x}}_i, \boldsymbol{w}_i))}$



线性化: $\tilde{\boldsymbol{x}}_{i+1} \simeq \boldsymbol{F}_{\tilde{x}} \tilde{\boldsymbol{x}}_i + \boldsymbol{F}_{\boldsymbol{w}} \boldsymbol{w}_i$

线性化于 $\tilde{x}_i = \mathbf{0}, \boldsymbol{w}_i = \mathbf{0}$

其中 $\boldsymbol{F}_{\tilde{x}} = \left(\dfrac{\partial \boldsymbol{G}(\tilde{\boldsymbol{x}}_i, \boldsymbol{g}(\mathbf{0},\mathbf{0}))}{\partial \tilde{\boldsymbol{x}}_i} + \dfrac{\partial \boldsymbol{G}(\mathbf{0}, \boldsymbol{g}(\tilde{\boldsymbol{x}}_i,\mathbf{0}))}{\partial \boldsymbol{g}(\tilde{\boldsymbol{x}}_i,\mathbf{0})} \dfrac{\partial \boldsymbol{g}(\tilde{\boldsymbol{x}}_i,\mathbf{0})}{\partial \tilde{\boldsymbol{x}}_i} \right) \Bigg|_{\tilde{x}_i=\mathbf{0}}$ $\quad \boldsymbol{F}_{\boldsymbol{w}} = \left(\dfrac{\partial \boldsymbol{G}(\mathbf{0}, \boldsymbol{g}(\mathbf{0},\boldsymbol{w}_i))}{\partial \boldsymbol{g}(\mathbf{0},\boldsymbol{w}_i)} \dfrac{\partial \boldsymbol{g}(\mathbf{0},\boldsymbol{w}_i)}{\partial \boldsymbol{w}_i} \right) \Bigg|_{\boldsymbol{w}_i=\mathbf{0}}$

$$\boldsymbol{G}(\tilde{\boldsymbol{x}}_i, \boldsymbol{g}(\tilde{\boldsymbol{x}}_i, \boldsymbol{w}_i)) = \left(\left(\hat{\boldsymbol{x}}_i \boxplus \tilde{\boldsymbol{x}}_i\right) \boxplus \boldsymbol{g}\left(\tilde{\boldsymbol{x}}_i, \boldsymbol{w}_i\right)\right) \boxminus \left(\hat{\boldsymbol{x}}_i \boxplus \boldsymbol{g}\left(\mathbf{0},\mathbf{0}\right)\right) := \left((\boldsymbol{a} \boxplus \boldsymbol{b}) \boxplus \boldsymbol{c}\right) \boxminus \boldsymbol{d} \qquad \boldsymbol{G}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}^{18}, \boldsymbol{a}, \boldsymbol{d} \in SO(3) \times \mathbb{R}^{15}$$

Case 1: $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d} \in \mathbb{R}^n \qquad \dfrac{\partial \boldsymbol{G}}{\partial \boldsymbol{b}} = \dfrac{\partial \boldsymbol{G}}{\partial \boldsymbol{c}} = \boldsymbol{I}_n$

Case 2: $\boldsymbol{G}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}^3, \boldsymbol{a}, \boldsymbol{d} \in SO(3) \qquad \dfrac{\partial \boldsymbol{G}}{\partial \boldsymbol{b}} = \boldsymbol{A}(\boldsymbol{G})^{-T} \operatorname{Exp}(-\boldsymbol{c}) \boldsymbol{A}(\boldsymbol{b})^T, \dfrac{\partial \boldsymbol{G}}{\partial \boldsymbol{c}} = \boldsymbol{A}(\boldsymbol{G})^{-T} \boldsymbol{A}(\boldsymbol{c})^T$

其中 $\boldsymbol{A}(\boldsymbol{u})^{-1} = \boldsymbol{I} - \dfrac{1}{2} \lfloor \boldsymbol{u} \rfloor_\wedge + \left(1 - \alpha\left(\|\boldsymbol{u}\|\right)\right) \dfrac{\lfloor \boldsymbol{u} \rfloor_\wedge^2}{\|\boldsymbol{u}\|^2} \qquad \alpha(m) = \dfrac{m}{2} \cot\left(\dfrac{m}{2}\right) = \dfrac{m}{2} \dfrac{\cos(m/2)}{\sin(m/2)}$

(左雅可比矩阵，源于 Baker-Campbell-Hausdorff (BCH) 公式)

2.2 前向传播

线性化: $$\tilde{\boldsymbol{x}}_{i+1} \simeq \boldsymbol{F}_{\tilde{x}} \tilde{\boldsymbol{x}}_i + \boldsymbol{F}_{\boldsymbol{w}} \boldsymbol{w}_i$$

线性化于 $\tilde{\boldsymbol{x}}_i = \boldsymbol{0}, \boldsymbol{w}_i = \boldsymbol{0}$

其中 $$\boldsymbol{F}_{\tilde{x}} = \left(\frac{\partial \boldsymbol{G}(\tilde{\boldsymbol{x}}_i, \boldsymbol{g}(\boldsymbol{0},\boldsymbol{0}))}{\partial \tilde{\boldsymbol{x}}_i} + \frac{\partial \boldsymbol{G}(\boldsymbol{0}, \boldsymbol{g}(\tilde{\boldsymbol{x}}_i, \boldsymbol{0}))}{\partial \boldsymbol{g}(\tilde{\boldsymbol{x}}_i, \boldsymbol{0})} \frac{\partial \boldsymbol{g}(\tilde{\boldsymbol{x}}_i, \boldsymbol{0})}{\partial \tilde{\boldsymbol{x}}_i} \right) \Bigg|_{\tilde{\boldsymbol{x}}_i = \boldsymbol{0}} \qquad \boldsymbol{F}_{\boldsymbol{w}} = \left(\frac{\partial \boldsymbol{G}(\boldsymbol{0}, \boldsymbol{g}(\boldsymbol{0}, \boldsymbol{w}_i))}{\partial \boldsymbol{g}(\boldsymbol{0}, \boldsymbol{w}_i)} \frac{\partial \boldsymbol{g}(\boldsymbol{0}, \boldsymbol{w}_i)}{\partial \boldsymbol{w}_i} \right) \Bigg|_{\boldsymbol{w}_i = \boldsymbol{0}}$$

$$\boldsymbol{G}(\tilde{\boldsymbol{x}}_i, \boldsymbol{g}(\tilde{\boldsymbol{x}}_i, \boldsymbol{w}_i)) = \left(\left(\hat{\boldsymbol{x}}_i \boxplus \tilde{\boldsymbol{x}}_i\right) \boxplus \boldsymbol{g}\left(\tilde{\boldsymbol{x}}_i, \boldsymbol{w}_i\right)\right) \boxminus \left(\hat{\boldsymbol{x}}_i \boxplus \boldsymbol{g}\left(\boldsymbol{0},\boldsymbol{0}\right)\right) := \left((\boldsymbol{a} \boxplus \boldsymbol{b}) \boxplus \boldsymbol{c}\right) \boxminus \boldsymbol{d} \qquad \boldsymbol{G}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}^{18}, \boldsymbol{a}, \boldsymbol{d} \in SO(3) \times \mathbb{R}^{15}$$

求 $\frac{\partial G}{\partial b}$:

$G = ((a \boxplus b) \boxplus c) \boxminus d = \mathrm{Log}\left(d^{-1}((a \boxplus b) \boxplus c)\right)$

则 $\mathrm{Exp}(G) = d^{-1} a \,\mathrm{Exp}(b)\,\mathrm{Exp}(c)$

$\mathrm{Exp}(G + \underline{\Delta G}) = d^{-1} a\,\mathrm{Exp}(b + \underline{\Delta b})\,\mathrm{Exp}(c)$ $\Rightarrow$ 目标: $\frac{\Delta G}{\Delta b} = ?$

$\mathrm{Exp}(G)\,\mathrm{Exp}(A(G)^T \Delta G) = d^{-1} a\,\mathrm{Exp}(b)\,\mathrm{Exp}(A(b)^T \Delta b)\,\mathrm{Exp}(c)$ $\Rightarrow$ BCH 近似公式

$$\boxed{\begin{aligned} A(u)^{-1} &= I - \tfrac{1}{2}\hat{u} + (1 - \alpha(\|u\|)) \frac{\hat{u}^2}{\|u\|^2} \\ \alpha(m) &= \tfrac{m}{2}\cot\left(\tfrac{m}{2}\right) \end{aligned}}$$

则 $\mathrm{Exp}(A(G)^T \Delta G) = \mathrm{Exp}^{-1}(c)\,\cancel{\mathrm{Exp}^{-1}(b)}\,\cancel{a^{-1}}\,\cancel{d}\,\cancel{d^{-1}}\,\cancel{a}\,\cancel{\mathrm{Exp}(b)}\,\mathrm{Exp}(A(b)^T \Delta b)\,\mathrm{Exp}(c)$

$= \mathrm{Exp}^T(c)\,\mathrm{Exp}(A(b)^T \Delta b)\,\mathrm{Exp}(c)$ 性质 $(Rp)^\wedge = R\hat{p}R^T$

$= \mathrm{Exp}\left(\mathrm{Exp}(-c)\,A(b)^T \Delta b\right)$

$\mathrm{Log}(\cdot)$

$\Rightarrow A(G)^T \Delta G = \mathrm{Exp}(-c)\,A(b)^T \Delta b \Rightarrow \frac{\Delta G}{\Delta b} = A(G)^{-T}\,\mathrm{Exp}(-c)\,A(b)^T$

求 $\frac{\partial G}{\partial c}$ 有

$\mathrm{Exp}(G + \Delta G) = d^{-1} a\,\mathrm{Exp}(b)\,\mathrm{Exp}(c + \Delta c)$

$\mathrm{Exp}(G)\,\mathrm{Exp}(A(G)^T \Delta G) = d^{-1} a\,\mathrm{Exp}(b)\,\mathrm{Exp}(c)\,\mathrm{Exp}(A(c)^T \Delta c)$

有 $\mathrm{Exp}(A(G)^T \Delta G) = \cancel{\mathrm{Exp}^{-1}(c)}\,\cancel{\mathrm{Exp}^{-1}(b)}\,\cancel{a^{-1}}\,\cancel{d}\,\cancel{d^{-1}}\,\cancel{a}\,\cancel{\mathrm{Exp}(b)}\,\cancel{\mathrm{Exp}(c)}\,\mathrm{Exp}(A(c)^T \Delta c)$

$\Rightarrow A(G)^T \Delta G = A(c)^T \Delta c \Rightarrow \frac{\Delta G}{\Delta c} = A(G)^{-T} A(c)^T$

线性化:

$$\tilde{\boldsymbol{x}}_{i+1} \simeq \boldsymbol{F}_{\tilde{x}}\tilde{\boldsymbol{x}}_i + \boldsymbol{F}_{\boldsymbol{w}}\boldsymbol{w}_i$$

$$\boldsymbol{F}_{\tilde{x}} = \left(\frac{\partial G(\tilde{\boldsymbol{x}}_i, \boldsymbol{g}(\boldsymbol{0},\boldsymbol{0}))}{\partial \tilde{\boldsymbol{x}}_i} + \frac{\partial G(\boldsymbol{0}, \boldsymbol{g}(\tilde{\boldsymbol{x}}_i,\boldsymbol{0}))}{\partial \boldsymbol{g}(\tilde{\boldsymbol{x}}_i,\boldsymbol{0})}\frac{\partial \boldsymbol{g}(\tilde{\boldsymbol{x}}_i,\boldsymbol{0})}{\partial \tilde{\boldsymbol{x}}_i}\right)\Bigg|_{\tilde{\boldsymbol{x}}_i=\boldsymbol{0}} \qquad \boldsymbol{F}_{\boldsymbol{w}} = \left(\frac{\partial G(\boldsymbol{0}, \boldsymbol{g}(\boldsymbol{0},\boldsymbol{w}_i))}{\partial \boldsymbol{g}(\boldsymbol{0},\boldsymbol{w}_i)}\frac{\partial \boldsymbol{g}(\boldsymbol{0},\boldsymbol{w}_i)}{\partial \boldsymbol{w}_i}\right)\Bigg|_{\boldsymbol{w}_i=\boldsymbol{0}}$$

$G|_{\tilde{x},\tilde{w}=0}$　　$g(\tilde{x},\tilde{\omega})$

仅前三维

$$\text{则}\ \frac{\partial G(\tilde{x}, g(0,0))}{\partial \tilde{x}} = \begin{bmatrix} \underbrace{A(0)^{-1}}_{=I} Exp(-g(0,0)) \underbrace{A(0)^T}_{=I} & 0 \\ 0 & I_{15} \end{bmatrix} = \begin{bmatrix} Exp(-f(\tilde{x},u,0)\Delta t) & 0 \\ 0 & I_{15} \end{bmatrix}$$

后15维为 R^n

无噪声, $n_w=0$

$$= \begin{bmatrix} Exp(-(\omega_m - \hat{b}_\omega)\Delta t) & 0 \\ 0 & I_{15\times15} \end{bmatrix}$$

$$\frac{\partial \mathbf{G}(\mathbf{0}, \mathbf{g}(\mathbf{0},\mathbf{w}))}{\partial \mathbf{g}(\mathbf{0},\mathbf{w})}\Big|_{\mathbf{w}=\mathbf{0}} = \frac{\partial \mathbf{G}(\mathbf{0}, \mathbf{g}(\widetilde{\mathbf{x}},\mathbf{0}))}{\partial \mathbf{g}(\widetilde{\mathbf{x}},\mathbf{0})}\Big|_{\widetilde{\mathbf{x}}=\mathbf{0}} = \begin{bmatrix} \mathbf{A}(\mathbf{0})^{-T}\mathbf{A}(\mathbf{g}(\mathbf{0},\mathbf{0}))^T & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{15\times15} \end{bmatrix}$$
$$= \begin{bmatrix} \mathbf{A}((\boldsymbol{\omega}_m - \widehat{\mathbf{b}}_\omega)\Delta t)^T & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{15\times15} \end{bmatrix}$$

由

$$\mathbf{g}(\widetilde{\mathbf{x}},\mathbf{w}) = \mathbf{f}(\mathbf{x},\mathbf{u},\mathbf{w})\Delta t = \begin{bmatrix} \boldsymbol{\omega}_m - \mathbf{b}_\omega - \mathbf{n}_\omega \\ {}^G\mathbf{v}_I \\ {}^G\mathbf{R}_I(\mathbf{a}_m - \mathbf{b}_\mathbf{a} - \mathbf{n}_\mathbf{a}) + {}^G\mathbf{g} \\ \mathbf{n}_{\mathbf{b}_\omega} \\ \mathbf{n}_{\mathbf{b}_\mathbf{a}} \\ \mathbf{0}_{3\times1} \end{bmatrix}\Delta t$$

$$= \begin{bmatrix} \boldsymbol{\omega}_m - \widehat{\mathbf{b}}_\omega - \widetilde{\mathbf{b}}_\omega - \mathbf{n}_\omega \\ {}^G\widehat{\mathbf{v}}_I + {}^G\widetilde{\mathbf{v}}_I \\ {}^G\widehat{\mathbf{R}}_I\mathrm{Exp}(\delta\boldsymbol{\theta}^T)\left(\mathbf{a}_m - \widehat{\mathbf{b}}_\mathbf{a} - \widetilde{\mathbf{b}}_\mathbf{a} - \mathbf{n}_\mathbf{a}\right) + {}^G\mathbf{g} \\ \mathbf{n}_{\mathbf{b}_\omega} \\ \mathbf{n}_{\mathbf{b}_\mathbf{a}} \\ \mathbf{0}_{3\times1} \end{bmatrix}\Delta t$$

得

$$\frac{\partial \mathbf{g}(\widetilde{\mathbf{x}},\mathbf{0})}{\partial \widetilde{\mathbf{x}}}\Big|_{\widetilde{\mathbf{x}}=\mathbf{0}} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_{3\times3}\Delta t & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{3\times3}\Delta t & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -{}^G\widehat{\mathbf{R}}_I\lfloor\mathbf{a}_m - \widehat{\mathbf{b}}_\mathbf{a}\rfloor_\wedge\Delta t & \mathbf{0} & \mathbf{0} & \mathbf{0} & -{}^G\widehat{\mathbf{R}}_I\Delta t & \mathbf{I}_{3\times3}\Delta t \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}$$

$$\frac{\partial \mathbf{g}(\mathbf{0},\mathbf{w})}{\partial \mathbf{w}}\Big|_{\mathbf{w}=\mathbf{0}} = \begin{pmatrix} -\mathbf{I}_{3\times3}\Delta t & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -{}^G\widehat{\mathbf{R}}_I\Delta t & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{3\times3} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{3\times3} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}$$

线性化: $\tilde{\boldsymbol{x}}_{i+1} \simeq \boldsymbol{F}_{\tilde{x}} \tilde{\boldsymbol{x}}_i + \boldsymbol{F}_w \boldsymbol{w}_i$

得到

$$\mathbf{F}_{\tilde{\mathbf{x}}} = \begin{bmatrix} \mathrm{Exp}\left(-\widehat{\boldsymbol{\omega}}_i \Delta t\right) & \mathbf{0} & \mathbf{0} & -\mathbf{A}(\widehat{\boldsymbol{\omega}}_i \Delta t)^T \Delta t & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{I}\Delta t & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -{}^G\widehat{\mathbf{R}}_{I_i} \lfloor \widehat{\mathbf{a}}_i \rfloor_\wedge \Delta t & \mathbf{0} & \mathbf{I} & \mathbf{0} & -{}^G\widehat{\mathbf{R}}_{I_i} \Delta t & \mathbf{I}\Delta t \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix},$$

$$\mathbf{F}_{\mathbf{w}} = \begin{bmatrix} -\mathbf{A}\left(\widehat{\boldsymbol{\omega}}_i \Delta t\right)^T \Delta t & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -{}^G\widehat{\mathbf{R}}_{I_i} \Delta t & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}\Delta t & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}\Delta t \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

协方差: $\hat{\boldsymbol{P}}_{i+1} = \boldsymbol{F}_{\tilde{x}} \hat{\boldsymbol{P}}_i \boldsymbol{F}_{\tilde{x}}^T + \boldsymbol{F}_w \boldsymbol{Q} \boldsymbol{F}_w^T ; \hat{\boldsymbol{P}}_0 = \overline{\boldsymbol{P}}_{k-1}.$ $\hat{\boldsymbol{P}}_k$ 是 $\boldsymbol{x}_k \boxminus \hat{\boldsymbol{x}}_k$ 的协方差

$\boldsymbol{Q}$ 是 $\boldsymbol{w}$ 的协方差

$${}^{I_k}\breve{\boldsymbol{p}}_{I_{j-1}} = {}^{I_k}\breve{\boldsymbol{p}}_{I_j} - {}^{I_k}\breve{\boldsymbol{v}}_{I_j}\Delta t, \quad \text{s.f.}\ {}^{I_k}\breve{\boldsymbol{p}}_{I_m} = \mathbf{0};$$

(s.f. : starting from)

$${}^{I_k}\breve{\boldsymbol{v}}_{I_{j-1}} = {}^{I_k}\breve{\boldsymbol{v}}_{I_j} - {}^{I_k}\breve{\boldsymbol{R}}_{I_j}(\boldsymbol{a}_{m_{i-1}} - \hat{\boldsymbol{b}}_{\boldsymbol{a}_k})\Delta t - {}^{I_k}\hat{\boldsymbol{g}}_k\Delta t,$$

$$\text{s.f.}\ {}^{I_k}\breve{\boldsymbol{v}}_{I_m} = {}^{G}\hat{\boldsymbol{R}}_{I_k}^{T}\,{}^{G}\hat{\boldsymbol{v}}_{I_k},\ {}^{I_k}\hat{\boldsymbol{g}}_k = {}^{G}\hat{\boldsymbol{R}}_{I_k}^{T}\,{}^{G}\hat{\boldsymbol{g}}_k;$$

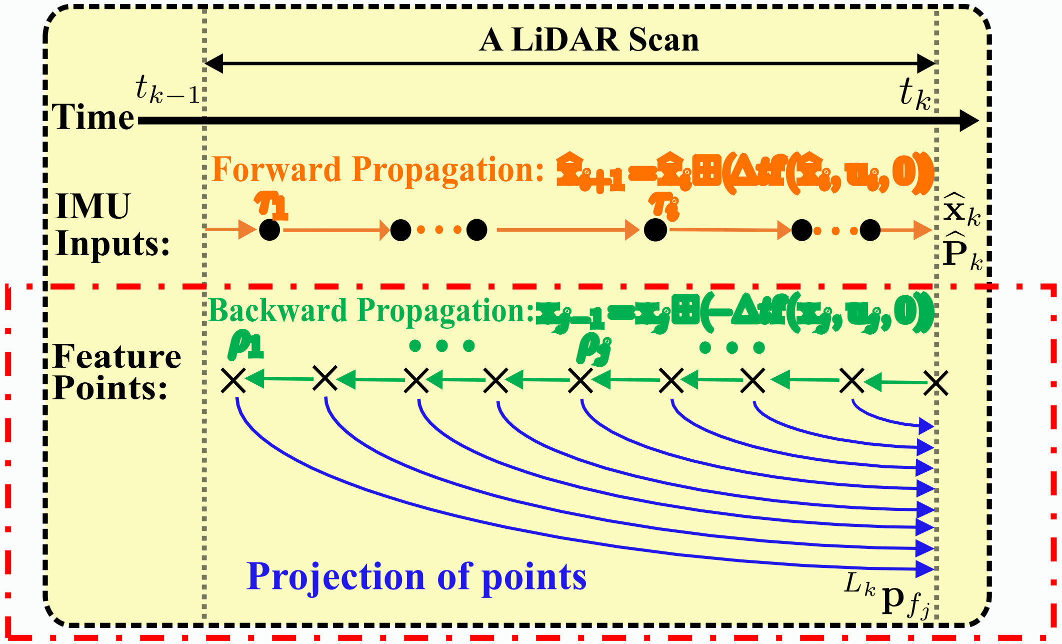
$${}^{I_k}\breve{\boldsymbol{R}}_{I_{j-1}} = {}^{I_k}\breve{\boldsymbol{R}}_{I_j}\operatorname{Exp}((\hat{\boldsymbol{b}}_{\boldsymbol{\omega}_k} - \boldsymbol{\omega}_{m_{i-1}})\Delta t), \text{s.f.}\ {}^{I_k}\boldsymbol{R}_{I_m} = \boldsymbol{I}$$

I_j : 第 j **个雷达点来临时** IMU 坐标系 $j = 1, \dots, m$

I_k : : 第 k **个雷达帧结束时** IMU坐标系

重要关系: $\boxed{I_m = I_k \Rightarrow {}^{I_k}\breve{\boldsymbol{p}}_{I_m} = \mathbf{0}, \quad {}^{I_k}\boldsymbol{R}_{I_m} = \boldsymbol{I}}$

→ 牵连速度仅有旋转速度
(最后一个雷达点到达时无平移量)



得到两个 IMU 系的相对位姿: $\boxed{{}^{I_k}\breve{\boldsymbol{T}}_{I_j} = ({}^{I_k}\breve{\boldsymbol{R}}_{I_j}, {}^{I_k}\breve{\boldsymbol{p}}_{I_j})}$

将相对于局部坐标系的测量对齐到一帧结束时的坐标系:

$${}^{L_k}\boldsymbol{p}_{f_j} = {}^{I}\boldsymbol{T}_L^{-1}\ {}^{I_k}\breve{\boldsymbol{T}}_{I_j}\ {}^{I}\boldsymbol{T}_L\ {}^{L_j}\boldsymbol{p}_{f_j}$$

相对于帧结束坐标系的测量 | 从 IMU 到 LiDAR 的外参矩阵（相对位姿）| 两时刻 IMU 系的相对位姿 | 从 LiDAR 到 IMU 的外参矩阵（相对位姿）| 相对于局部坐标系的测量

假设迭代 Kalman 滤波的迭代数为 κ $\kappa = 0, \hat{\boldsymbol{x}}_k^{\kappa} = \hat{\boldsymbol{x}}_k$

上述去畸变的雷达点可以进一步变换到世界坐标系： ${}^{G}\hat{\boldsymbol{p}}_{f_j}^{\kappa} = {}^{G}\hat{\boldsymbol{T}}_{I_k}^{\kappa}\,{}^{I}\boldsymbol{T}_L\,{}^{L_k}\boldsymbol{p}_{f_j}, \quad j = 1, \ldots, m.$

残差构建：某个特征点应该属于其周围特征点所确定的最近**平面或边。**

残差：

$$\boldsymbol{z}_j^{\kappa} = \boldsymbol{G}_j\left({}^{G}\hat{\boldsymbol{p}}_{f_j}^{\kappa} - {}^{G}\boldsymbol{q}_j\right)$$

$$\boldsymbol{G}_j = \begin{cases} \boldsymbol{u}_j^T, \text{平面的法向量 (对于平面特征)} \\ \lfloor \boldsymbol{u}_j \rfloor_{\wedge}, \text{边的方向 (对于边特征)} \end{cases}$$

真实点：

$${}^{L_j}\boldsymbol{p}_{f_j}^{\mathrm{gt}} = {}^{L_j}\boldsymbol{p}_{f_j} - {}^{L_j}\boldsymbol{n}_{f_j}$$

雷达点云的噪声

去畸变并投影到世界坐标系后，
真实点所对应的残差应该为 0:

$$\underbrace{\boldsymbol{G}_j\left({}^{G}\boldsymbol{T}_{I_k}\,{}^{I_k}\breve{\boldsymbol{T}}_{I_j}\,{}^{I}\boldsymbol{T}_L\left({}^{L_j}\boldsymbol{p}_{f_j} - {}^{L_j}\boldsymbol{n}_{f_j}\right) - {}^{G}\boldsymbol{q}_j\right)}_{\boldsymbol{h}_j\left(\boldsymbol{x}_k, {}^{L_j}\boldsymbol{n}_{f_j}\right)} = \boldsymbol{0}$$

观测和先验的线性化：

$$\begin{aligned}
\mathbf{0} &= \boldsymbol{h}_j\left(\mathbf{x}_k, {}^{L_j}\boldsymbol{n}_{f_j}\right) = \boldsymbol{h}_j\left(\hat{\mathbf{x}}_k^\kappa \boxplus \tilde{\mathbf{x}}_k^\kappa, {}^{L_j}\boldsymbol{n}_{f_j}\right) \\
&\simeq \boldsymbol{h}_j\left(\hat{\mathbf{x}}_k^\kappa, \mathbf{0}\right) + \boldsymbol{H}_j^\kappa \tilde{\mathbf{x}}_k^\kappa + \boldsymbol{v}_j \\
&= \boldsymbol{z}_j^\kappa + \boldsymbol{H}_j^\kappa \tilde{\mathbf{x}}_k^\kappa + \boldsymbol{v}_j
\end{aligned}$$

一阶泰勒展开

$\boldsymbol{v}_j \in \mathcal{N}(\mathbf{0}, \boldsymbol{R}_j)$ 来自雷达点测量误差

$$\mathbf{x}_k \boxminus \hat{\mathbf{x}}_k = \left(\hat{\mathbf{x}}_k^\kappa \boxplus \tilde{\mathbf{x}}_k^\kappa\right) \boxminus \hat{\mathbf{x}}_k = \hat{\mathbf{x}}_k^\kappa \boxminus \hat{\mathbf{x}}_k + \boldsymbol{J}^\kappa \tilde{\mathbf{x}}_k^\kappa$$

一阶泰勒展开

$\boldsymbol{J}^\kappa$ 可以通过右边公式计算

$$\boldsymbol{G}(\tilde{\boldsymbol{x}}_i, \boldsymbol{g}(\tilde{\boldsymbol{x}}_i, \boldsymbol{w}_i)) = \left(\left(\hat{\boldsymbol{x}}_i \boxplus \tilde{\boldsymbol{x}}_i\right) \boxplus \boldsymbol{g}\left(\tilde{\boldsymbol{x}}_i, \boldsymbol{w}_i\right)\right) \boxminus \left(\hat{\boldsymbol{x}}_i \boxplus \boldsymbol{g}\left(\mathbf{0}, \mathbf{0}\right)\right) := \left((\boldsymbol{a} \boxplus \boldsymbol{b}) \boxplus \boldsymbol{c}\right) \boxminus \boldsymbol{d} \qquad \boldsymbol{G}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}^{18}, \boldsymbol{a}, \boldsymbol{d} \in SO(3) \times \mathbb{R}^{15}$$

Case 1: $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d} \in \mathbb{R}^n \qquad \dfrac{\partial \boldsymbol{G}}{\partial \boldsymbol{b}} = \dfrac{\partial \boldsymbol{G}}{\partial \boldsymbol{c}} = \boldsymbol{I}_n$

Case 2: $\boldsymbol{G}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}^3, \boldsymbol{a}, \boldsymbol{d} \in SO(3) \qquad \dfrac{\partial \boldsymbol{G}}{\partial \boldsymbol{b}} = \boldsymbol{A}(\boldsymbol{G})^{-T} \operatorname{Exp}(-\boldsymbol{c}) \boldsymbol{A}(\boldsymbol{b})^T, \dfrac{\partial \boldsymbol{G}}{\partial \boldsymbol{c}} = \boldsymbol{A}(\boldsymbol{G})^{-T} \boldsymbol{A}(\boldsymbol{c})^T$

where $\boldsymbol{A}(\boldsymbol{u})^{-1} = \boldsymbol{I} - \dfrac{1}{2}\lfloor \boldsymbol{u} \rfloor_\wedge + \left(1 - \alpha\left(\|\boldsymbol{u}\|\right)\right) \dfrac{\lfloor \boldsymbol{u} \rfloor_\wedge^2}{\|\boldsymbol{u}\|^2} \qquad \alpha(m) = \dfrac{m}{2} \cot\left(\dfrac{m}{2}\right) = \dfrac{m}{2} \dfrac{\cos(m/2)}{\sin(m/2)}$

(右雅可比矩阵，源于 Baker-Campbell-Hausdorff (BCH) 公式)

$$\boldsymbol{H}_j^\kappa = \frac{\partial \boldsymbol{h}_j}{\partial \boldsymbol{p}_j} \frac{\partial \boldsymbol{p}_j}{\partial \tilde{\mathbf{x}}_j^\kappa} = \left[-\boldsymbol{G}_j {}^G\hat{\boldsymbol{R}}_I^\kappa \left\lfloor {}^I\boldsymbol{T}_L {}^L\boldsymbol{p}_j \right\rfloor_\wedge, \boldsymbol{G}_j, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}\right]$$

将先验和观测结合起来可得到下述

极大后验估计（maximum a posteriori estimate，MAP）：

$$\min_{\tilde{\boldsymbol{x}}_k^{\kappa}} \left(\left\| \boldsymbol{x}_k \boxminus \hat{\boldsymbol{x}}_k \right\|^2_{\hat{\boldsymbol{P}}_k^{-1}} + \sum_{j=1}^{m} \left\| \boldsymbol{z}_j^{\kappa} + \boldsymbol{H}_j^{\kappa} \tilde{\boldsymbol{x}}_k^{\kappa} \right\|^2_{\boldsymbol{R}_j^{-1}} \right)$$

令 $\boldsymbol{H} = [(\boldsymbol{H}_1^{\kappa})^T, \dots, (\boldsymbol{H}_m^{\kappa})^T]^T, \boldsymbol{R} = \mathrm{diag}(\boldsymbol{R}_1, \dots, \boldsymbol{R}_m), \boldsymbol{P} = (\boldsymbol{J}^{\kappa})^{-1} \hat{\boldsymbol{P}}_k (\boldsymbol{J}^{\kappa})^{-T},\ \boldsymbol{z}_k^{\kappa} = [\boldsymbol{z}_1^{\kappa T}, \dots, \boldsymbol{z}_m^{\kappa T}]^T$

迭代 Kalman 滤波器:

$$\boldsymbol{K} = \boldsymbol{P}\boldsymbol{H}^T (\boldsymbol{H}\boldsymbol{P}\boldsymbol{H}^T + \boldsymbol{R})^{-1},$$

$$\hat{\boldsymbol{x}}_k^{\kappa+1} = \hat{\boldsymbol{x}}_k^{\kappa} \boxplus \left(-\boldsymbol{K}\boldsymbol{z}_k^{\kappa} - (\boldsymbol{I} - \boldsymbol{K}\boldsymbol{H})(\boldsymbol{J}^{\kappa})^{-1} \left(\hat{\boldsymbol{x}}_k^{\kappa} \boxminus \hat{\boldsymbol{x}}_k \right) \right).$$

更新后的估计又被用于计算残差，不断重复直至收敛: $\left\| \hat{\boldsymbol{x}}_k^{\kappa+1} \boxminus \hat{\boldsymbol{x}}_k^{\kappa} \right\| < \epsilon$

收敛后，最优估计状态和协方差矩阵为: $\bar{\boldsymbol{x}}_k = \hat{\boldsymbol{x}}_k^{\kappa+1}, \bar{\boldsymbol{P}}_k = \left(\boldsymbol{I} - \boldsymbol{K}\boldsymbol{H} \right) \boldsymbol{P}$

直接解优化问题（最小二乘）可得以下卡尔曼增益公式

$$\boldsymbol{K}=\underbrace{\left(\boldsymbol{H}^{T}\boldsymbol{R}^{-1}\boldsymbol{H}+\boldsymbol{P}^{-1}\right)^{-1}}_{\in\mathbb{R}^{18\times18}}\underbrace{\boldsymbol{H}^{T}}_{\in\mathbb{R}^{18\times3m}}\underbrace{\boldsymbol{R}^{-1}}_{\in\mathbb{R}^{3m\times3m}}.$$

需要求一个常数大小矩阵的逆 $\mathcal{O}(1)$

V.S. 原有形式：

$$\boldsymbol{K}=\underbrace{\boldsymbol{P}}_{\in\mathbb{R}^{18\times18}}\underbrace{\boldsymbol{H}^{T}}_{\in\mathbb{R}^{18\times3m}}\underbrace{\left(\boldsymbol{H}\boldsymbol{P}\boldsymbol{H}^{T}+\boldsymbol{R}\right)^{-1}}_{\in\mathbb{R}^{3m\times3m}}$$

需要求一个大小与观测点数正相关的巨型矩阵的逆 $\mathcal{O}(m^{3})$

两个公式的等价性可用下述 Woodbury 矩阵求逆引理验证：

$$\left(\boldsymbol{P}^{-1}+\boldsymbol{H}^{T}\boldsymbol{R}^{-1}\boldsymbol{H}\right)^{-1}=\boldsymbol{P}-\boldsymbol{P}\boldsymbol{H}^{T}\left(\boldsymbol{H}\boldsymbol{P}\boldsymbol{H}^{T}+\boldsymbol{R}\right)^{-1}\boldsymbol{H}\boldsymbol{P}$$

Substituting above into (20), we can get:

$$\begin{aligned}\mathbf{K}&=\left(\mathbf{H}^{T}\mathbf{R}^{-1}\mathbf{H}+\mathbf{P}^{-1}\right)^{-1}\mathbf{H}^{T}\mathbf{R}^{-1}\\&=\mathbf{P}\mathbf{H}^{T}\mathbf{R}^{-1}-\mathbf{P}\mathbf{H}^{T}\left(\mathbf{H}\mathbf{P}\mathbf{H}^{T}+\mathbf{R}\right)^{-1}\mathbf{H}\mathbf{P}\mathbf{H}^{T}\mathbf{R}^{-1}\end{aligned}$$

Now note that $\mathbf{H}\mathbf{P}\mathbf{H}^{T}\mathbf{R}^{-1}=(\mathbf{H}\mathbf{P}\mathbf{H}^{T}+\mathbf{R})\mathbf{R}^{-1}-\mathbf{I}$. Substituting it into above, we can get the standard Kalman gain formula in (18), as shown below.

$$\begin{aligned}\mathbf{K}&=\mathbf{P}\mathbf{H}^{T}\mathbf{R}^{-1}-\mathbf{P}\mathbf{H}^{T}\mathbf{R}^{-1}+\mathbf{P}\mathbf{H}^{T}\left(\mathbf{H}\mathbf{P}\mathbf{H}^{T}+\mathbf{R}\right)^{-1}\\&=\mathbf{P}\mathbf{H}^{T}\left(\mathbf{H}\mathbf{P}\mathbf{H}^{T}+\mathbf{R}\right)^{-1}.\end{aligned}$$

3 仿真结果

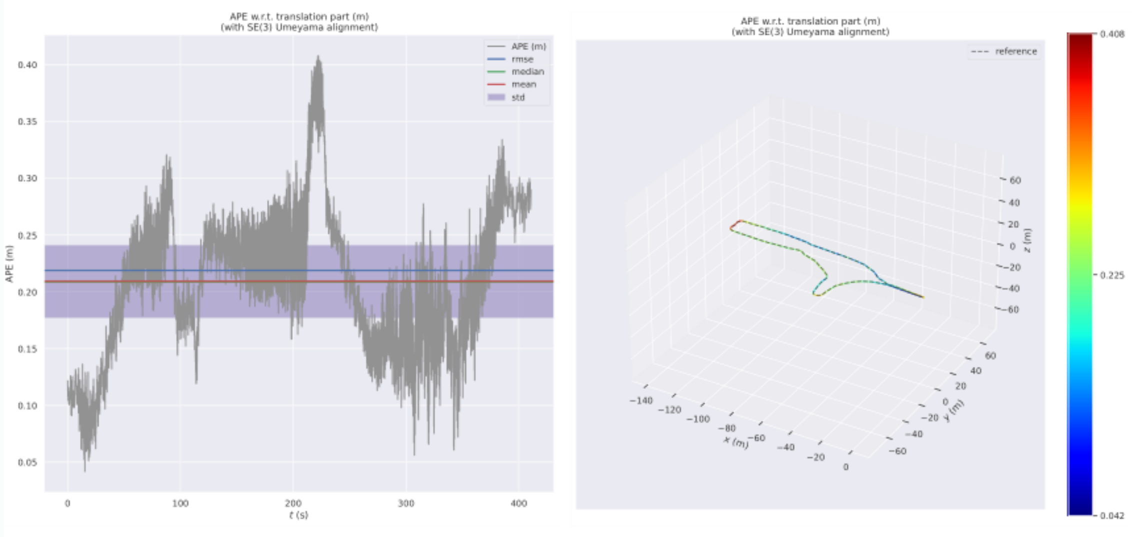
3. 仿真复现

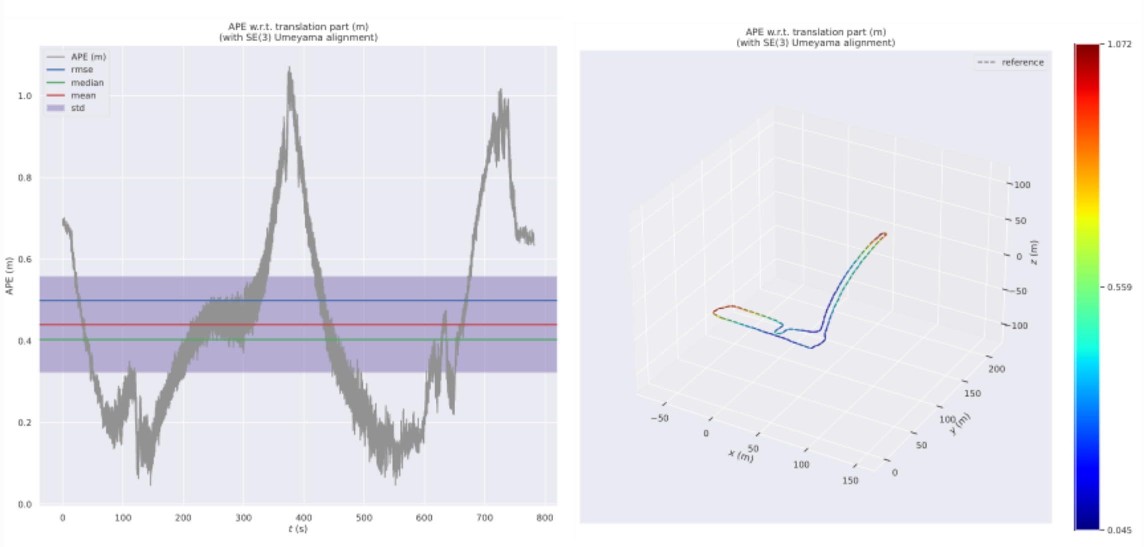
使用 **M3DGR dataset**[3] 进行复现。Bag 1（左图）为 Outdoor1，Bag 2（右图）为 Outdoor 4。

Table 1: Error statistics of the estimated trajectory.

	RMSE	Mean	Median	Std. Dev	Min	Max	SSE*
Bag 1	0.219	0.209	0.209	0.064	0.042	0.408	196.9
Bag 2	0.499	0.440	0.403	0.236	0.045	1.072	1945.0

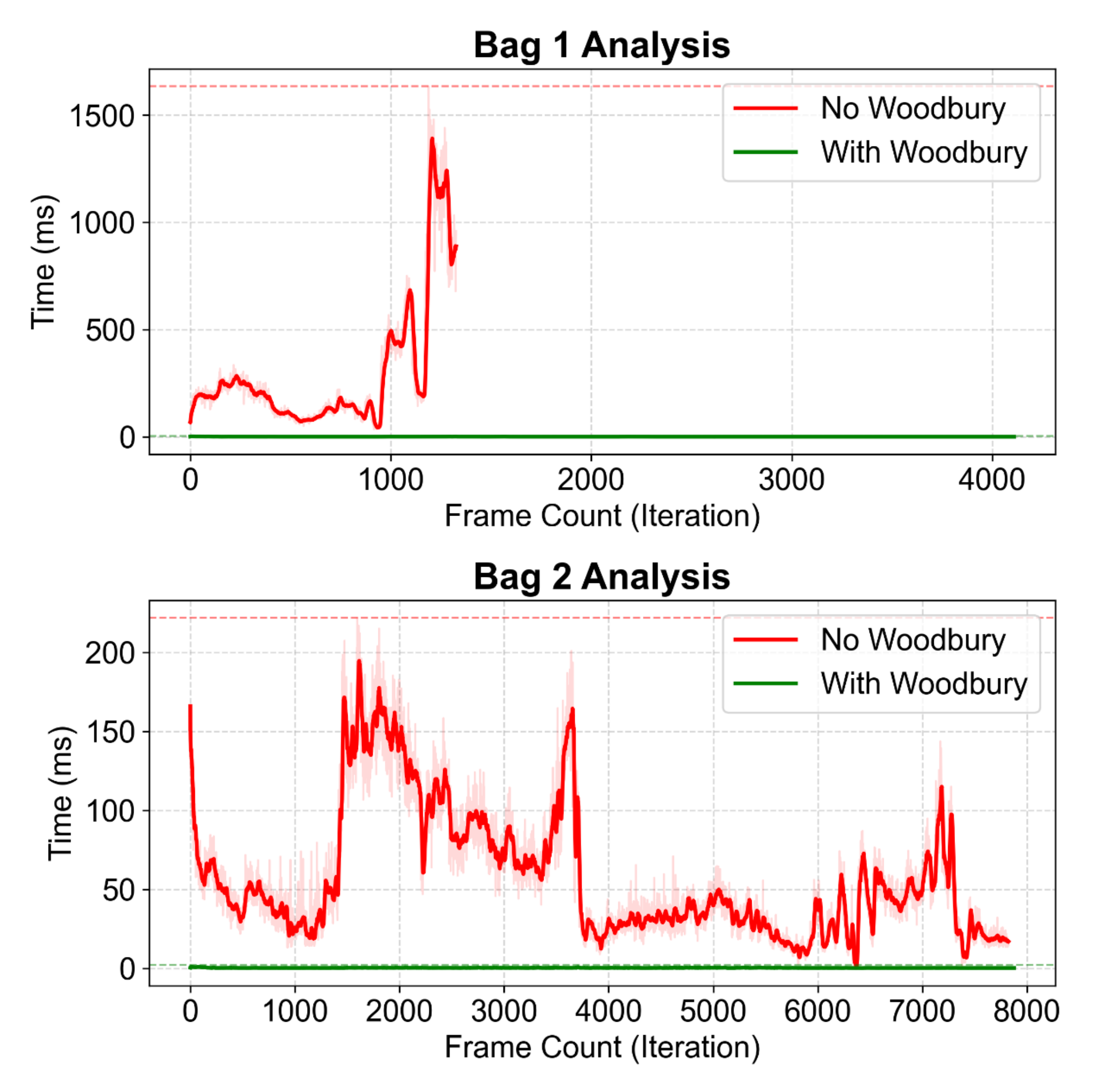
* SSE denotes the Sum of Squared Errors, i.e., SSE $= \sum_{k=1}^{N} \|\hat{\mathbf{p}}_k - \mathbf{p}_k^{\mathrm{gt}}\|^2$.





[3] Zhang, D., Zhang, J., Sun, Y., Li, T., Yin, H., Xie, H., & Yin, J. (2025). Towards robust sensor-fusion ground SLAM: A comprehensive benchmark and a resilient framework. arXiv preprint arXiv:2507.08364.

3. 仿真复现——两种卡尔曼增益公式的计算效率对比



Method	Bag 1		Bag 2	
	Max.(ms)	Avg.(ms)	Max.(ms)	Avg.(ms)
Old Formula	1633.65	305.35	221.78	57.48
New Formula	2.97	0.40	2.39	0.33

使用原本的 Kalman 增益公式会**显著降低 FAST-LIO 的处理速度。**Bag 1 没跑完；Bag 2 由于后续点比较稀疏，最后跑完了，但是仍然出现了很大的滞后。